JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **27**, No. 4, November 2014
http://dx.doi.org/10.14403/jcms.2014.27.4.553

DENSITIES FOR 4-RANKS OF REAL QUADRATIC FUNCTION FIELDS

Hwanyup Jung*

Abstract. In this paper we study of densities of the 4-rank of narrow ideal class groups of real quadratic function fields over the rational function field $\mathbb{F}_q(T)$ when $q \equiv 3 \bmod 4$.

1. Introduction and statement of result

Let $k = \mathbb{F}_q(T)$ be the rational function field over the finite field $\mathbb{F}_q$ of odd characteristic and ∞ be the prime of k associated to $1/T$, which is called the *infinite prime* of k. Let K be a quadratic extension of k. We say that K is *real* extension of k if ∞ splits and *imaginary* extension of k otherwise. In this paper, by a *real quadratic function field* we always mean a real quadratic extension of k. Wittmann [6] motivated by Gerth's article [2] has studied the distribution of the 4-rank of ideal class groups of (ramified) imaginary quadratic function fields K over k. The aim of this paper is to study the distribution of the 4-rank of (narrow) ideal class groups of real quadratic function fields K over k.

Let $\mathbb{A} = \mathbb{F}_q[T]$ be the polynomial ring. Let $\mathcal{D}$ be the subset of $\mathbb{A}$ consisting of all monic square free polynomials $D \neq 1$ in $\mathbb{A}$ of even degree. For any $D \in \mathcal{D}$, $k_D := k(\sqrt{D})$ is a real quadratic function field over k. Moreover, for any real quadratic function field K of k, there exists a unique $D \in \mathcal{D}$ such that $K = k_D$. Let $\mathcal{O}_D$ be the integral closure of $\mathbb{A}$ in k_D and $\mathcal{C}_D$ be the narrow ideal class group of $\mathcal{O}_D$. Let $r_4^+(D)$ be the 4-rank of $\mathcal{C}_D$, that is, $r_4^+(D) = \dim_{\mathbb{F}_2}(\mathcal{C}_D^2/\mathcal{C}_D^4)$. Let $\omega(D)$ denote the number of distinct monic irreducible divisors of D. For positive integers t, d (d is even) and a nonnegative integer r with $0 \leq r \leq t-1$, write $A_{t;d}$ for the set of all real quadratic function fields k_D with $D \in \mathcal{D}, \deg D = d$

Received May 27, 2014; Accepted October 06, 2014.
2010 Mathematics Subject Classification: Primary 11R29, 11R45, 11R58.
Key words and phrases: 4-rank of class groups, real quadratic function fields.
This work was supported by the research grant of Chungbuk National University in 2013.

and $\omega(D) = t$, and $A_{t,r;d}$ for the subset of $A_{t;d}$ consisting of all $k_D \in A_{t;d}$ with $r_4^+(D) = r$. We define a density

$$\alpha_t(r) = \lim_{\substack{d\to\infty\\ d:\text{even}}} \frac{|A_{t,r;d}|}{|A_{t;d}|}.$$

We also define the limit density

$$\alpha_\infty(r) = \lim_{t\to\infty} \alpha_t(r).$$

For any $0 < x < 1$ and $n \in \mathbb{N} \cup \{\infty\}$, we put $(x)_n = \prod_{i=1}^{n}(1 - x^i)$. The main result of this paper is the following theorem.

THEOREM 1.1. *Assume that $q \equiv 3 \bmod 4$. Then the limit density $\delta_\infty(r)$ exists for all $r \geq 0$, and we have*

$$\alpha_\infty(r) = 2^{-r(r+1)} \frac{(\frac{1}{2})_\infty}{(\frac{1}{2})_r(\frac{1}{2})_{r+1}}.$$

For small values of r, the limit density $\alpha_\infty(r)$ equals (up to 10 decimal digits):

$\delta_\infty(0)$	0.5775761902
$\delta_\infty(1)$	0.1925253967
$\delta_\infty(2)$	0.0068759070
$\delta_\infty(3)$	0.0000859488
$\delta_\infty(4)$	0.0000003032

2. Narrow genus field and Rèdei-matrix

2.1. Narrow ideal class group

For $D \in \mathcal{D}$, write $\mathcal{C}l_D = \mathcal{I}_D/\mathcal{P}_D$ for the ordinary ideal class group of $\mathcal{O}_D$, where $\mathcal{I}_D$ is the group of fractional ideals of $\mathcal{O}_D$ and $\mathcal{P}_D = \{(x) \in \mathcal{I}_D : x \in k_D^*\}$. Let k_∞ be the completion of k at ∞. Fix a sign map $sgn : k_\infty^* \to \mathbb{F}_q^*$ with $sgn(1/T) = 1$. Define $\overline{sgn}(x) = sgn(x)^{\frac{q-1}{2}}$ for any $x \in k_\infty^*$. Fix an embedding of k_D into k_∞. We say that an element $x \in k_D^*$ is *totally positive* if $\overline{sgn}(x) = \overline{sgn}(x^\sigma) = 1$, where σ is the generator of $G = Gal(k_D/k)$. Let k_D^+ be the subset of k_D^* consisting of all totally positive elements. The *narrow ideal class group* $\mathcal{C}_D$ of $\mathcal{O}_D$ is defined by $\mathcal{C}_D := \mathcal{I}_D/\mathcal{P}_D^+$, where $\mathcal{P}_D^+ = \{(x) \in \mathcal{I}_D : x \in k_D^+\}$. Let $\mathcal{N}$ be the norm map from k_D to k. For any fractional ideal $\mathfrak{a}$ of $\mathcal{O}_D$, we denote by $[\mathfrak{a}]$ and $[\mathfrak{a}]_+$ the images of $\mathfrak{a}$ in $\mathcal{C}l_D$ and $\mathcal{C}_D$, respectively.

Let $\pi : \mathcal{C}_D \to \mathcal{C}l_D$ be the canonical surjective homomorphism defined by $\pi([\mathfrak{a}]_+) = [\mathfrak{a}]$.

LEMMA 2.1. *We have* $|Ker(\pi)| \leq 2$, *and* $|Ker(\pi)| = 2$ *if and only if* $\mathcal{N}(E_D) = \mathbb{F}_q^{*2}$, *where* E_D *is the group of units of* $\mathcal{O}_D$.

Proof. Note that $Ker(\pi) = \mathcal{P}_D/\mathcal{P}_D^+$. There is an exact sequence

$$1 \to E_D/E_D^+ \to k_D^*/k_D^+ \to \mathcal{P}_D/\mathcal{P}_D^+ \to 1,$$

where $E_D^+ = k_D^+ \cap E_D$. Consider a homomorphism

$$\psi : k_D^* \to \{\pm 1\} \times \{\pm 1\},\ x \mapsto (\overline{sgn}(x), \overline{sgn}(x^\sigma)),$$

whose kernel is k_D^+. Let γ be a generator of $\mathbb{F}_q^*$. We have $\psi(\gamma) = (-1,-1)$. Let $x_D \in k_D$ be defined by

$$x_D = \begin{cases} \sqrt{D} & \text{if } q \equiv 3 \bmod 4, \\ A + B\sqrt{D} & \text{if } q \equiv 1 \bmod 4, \end{cases}$$

where $A, B \in \mathbb{A}$ are chosen to satisfy $\deg A = \deg B + \frac{1}{2}\deg D$ and $sgn(A)^2 - sgn(B)^2 \notin \mathbb{F}_q^{*2}$. If $q \equiv 3 \bmod 4$, then we have $\overline{sgn}(\mathcal{N}(x_D)) = sgn(-D)^{\frac{q-1}{2}} = -1$. If $q \equiv 1 \bmod 4$, then, since $sgn(A)^2 - sgn(B)^2 \notin \mathbb{F}_q^{*2}$, we have

$$\overline{sgn}(\mathcal{N}(x_D)) = (sgn(A)^2 - sgn(B)^2)^{\frac{q-1}{2}} = -1.$$

Hence $\psi(x_D) = (1,-1)$ or $(-1,1)$. Thus ψ is surjective, and it induces an isomorphism $k_D^*/k_D^+ \cong \{\pm 1\} \times \{\pm 1\}$. Let ε_D be the fundamental unit of k_D, i.e., $E_D = \mathbb{F}_q^* \times \langle \varepsilon_D \rangle$.

Assume that $\mathcal{N}(E_D) = \mathbb{F}_q^{*2}$. Then $\mathcal{N}(\varepsilon_D) \in \mathbb{F}_q^{*2}$ and so $\overline{sgn}(\varepsilon_D) = \overline{sgn}(\sigma(\varepsilon_D))$. If $\overline{sgn}(\varepsilon_D) = 1$, then $\varepsilon_D \in E_D^+$, so we have $E_D^+ = \mathbb{F}_q^{*2} \times \langle \varepsilon_D \rangle, E_D/E_D^+ \cong \mathbb{Z}/2\mathbb{Z}$ and $|Ker(\pi)| = 2$. If $\overline{sgn}(\varepsilon_D) = -1$, then $\gamma\varepsilon_D \in E_D^+$, so we have $\mathbb{F}_q^{*2} \times \langle \varepsilon_D^2 \rangle \subset E_D^+, E_D/E_D^+ \cong \mathbb{Z}/2\mathbb{Z}$ and $|Ker(\pi)| = 2$.

Now, we assume that $\mathcal{N}(E_D) = \mathbb{F}_q^*$. Then $\mathcal{N}(\varepsilon_D) \in \mathbb{F}_q^* \setminus \mathbb{F}_q^{*2}$. We may assume $\sigma(\varepsilon_D) = \gamma\varepsilon_D$. Then $\overline{sgn}(\varepsilon_D)\overline{sgn}(\sigma(\varepsilon_D)) = -1$, say $\overline{sgn}(\varepsilon_D) = 1$ and $\overline{sgn}(\sigma(\varepsilon_D)) = -1$. For any $\gamma^a\varepsilon_D^b \in E_D$, we have $\overline{sgn}(\gamma^a\varepsilon_D^b) = (-1)^a$ and $\overline{sgn}(\sigma(\gamma^a\varepsilon_D^b)) = (-1)^{a+b}$. Hence $\gamma^a\varepsilon_D^b \in E_D^+$ if and only if $a \equiv b \equiv 0 \bmod 2$. Thus $E_D^+ = \mathbb{F}_q^{*2} \times \langle \varepsilon_D^2 \rangle$, $E_D/E_D^+ \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $|Ker(\pi)| = 1$. □

Write $r_2(D)$ and $r_2^+(D)$ for the 2-ranks of $\mathcal{C}l_D$ and $\mathcal{C}_D$, respectively, i.e., $r_2(D) = \dim_{\mathbb{F}_2}(\mathcal{C}l_D/\mathcal{C}l_D^2)$ and $r_2^+(D) = \dim_{\mathbb{F}_2}(\mathcal{C}_D/\mathcal{C}_D^2)$. We say that

D is *special* if each monic irreducible divisor of D is of even degree. It is known [5, Theorem 2.1] that

$$(2.1)\qquad r_2(D) = \begin{cases} \omega(D) - 1 & \text{if } D \text{ is special,} \\ \omega(D) - 2 & \text{otherwise.} \end{cases}$$

LEMMA 2.2. *Let $D \in \mathcal{D}$ with monic irreducible factorization $D = P_1 \cdots P_t$. Let $\mathfrak{p}_i$ be the unique prime ideal of $\mathcal{O}_D$ lying above P_i for $1 \le i \le t$. Then $r_2^+(D) = t - 1$. Moreover, we have $\mathcal{C}_D^G = \langle [\mathfrak{p}_1]_+, \ldots, [\mathfrak{p}_t]_+ \rangle$, except in the case that $\mathcal{N}(E_D) = \mathbb{F}_q^{*2}$ and $q \equiv 1 \bmod 4$. In this exceptional case we have $\mathcal{C}_D^G = \langle [\mathfrak{p}_1]_+, \ldots, [\mathfrak{p}_t]_+, [\mathfrak{b}]_+ \rangle$, where $\mathfrak{b}$ is any principal ideal generated by an element $\beta \in k_D^* \setminus k_D^+$ such that $\varepsilon\beta \in k_D^* \setminus k_D^+$ for any $\varepsilon \in E_D$, and $[\mathfrak{b}]_+ \in \mathcal{C}_D^{1-\sigma}$ or $[\mathfrak{b}]_+ \notin \mathcal{C}_D^{1-\sigma}$ according as D is special or non-special.*

Proof. It is known [5, Corollary 2.4] that $\mathcal{C}l_D^G$ is generated by $[\mathfrak{p}_1], \ldots, [\mathfrak{p}_t]$, except the case that D is special and $\mathcal{N}(E_D) = \mathbb{F}_q^{*2}$, and in this exceptional case $\mathcal{C}l_D^G$ is generated by $[\mathfrak{p}_1], \ldots, [\mathfrak{p}_t], [\mathfrak{a}]$, where $\mathfrak{a}$ is a fractional ideal of $\mathcal{O}_D$ such that $\mathfrak{a}^{1-\sigma} = \alpha\mathcal{O}_D$ with $\alpha \in k_D^*$ satisfying $\mathcal{N}(\alpha) \in \mathbb{F}_q^* \setminus \mathbb{F}_q^{*2}$. First, we assume that $|Ker(\pi)| = 1$. Then $\mathcal{C}_D^G \cong \mathcal{C}l_D^G$ and $r_2^+(D) = r_2(D)$. Since $\mathcal{N}(E_D) = \mathbb{F}_q^*$, D is special, so $r_2^+(D) = r_2(D) = t - 1$ and $\mathcal{C}_D^G$ is generated by $[\mathfrak{p}_1]_+, \ldots, [\mathfrak{p}_t]_+$.

Now, we assume that $|Ker(\pi)| = 2$. In this case $\mathcal{N}(E_D) = \mathbb{F}_q^{*2}$. From the following commutative diagram

$$1[r]Ker(\pi)[r][d]^{1-\sigma}\mathcal{C}_D[r]^{\pi}[d]^{1-\sigma}\mathcal{C}l_D[r][d]^{1-\sigma}11[r]Ker(\pi)[r]\mathcal{C}_D[r]^{\pi}\mathcal{C}l_D[r]1$$

we get an exact sequence (by Snake Lemma)

$$1 \to Ker(\pi) \to \mathcal{C}_D^G \to \mathcal{C}l_D^G \xrightarrow{\Psi} Ker(\pi) \to \mathcal{C}_D/\mathcal{C}_D^{1-\sigma} \to \mathcal{C}l_D/\mathcal{C}l_D^{1-\sigma} \to 1,$$

where $\Psi([\mathfrak{b}]) = [\mathfrak{b}]_+^2 \in Ker(\pi)$ for any $[\mathfrak{b}] \in \mathcal{C}l_D^G$. Note that

$$(2.2)\qquad \Psi([\mathfrak{p}_i]) = [\mathfrak{p}_i^2]_+ = [(P_i)]_+ = 1$$

for $1 \le i \le t$. When $q \equiv 3 \bmod 4$, it can be easily shown that $\varepsilon\sqrt{D} \notin k_D^* \setminus k_D^+$ for any $\varepsilon \in E_D$. Hence, in this case, we have

$$(2.3)\qquad Ker(\pi) = \{1, [(\sqrt{D})]_+\} \text{ with } [(\sqrt{D})]_+ = [\mathfrak{p}_1]_+ \cdots [\mathfrak{p}_t]_+.$$

Case 1. D is special. Since $\mathcal{N}(\alpha) \in \mathbb{F}_q^* \setminus \mathbb{F}_q^{*2}$, we have $\alpha \in k_D^* \setminus k_D^+$. Moreover, it can be shown that $\varepsilon\alpha \in k_D^* \setminus k_D^+$ for any $\varepsilon \in E_D$, so we have

$$(2.4)\qquad \Psi([\mathfrak{a}]) = [\mathfrak{a}^2]_+ = [(\alpha)]_+ \ne 1.$$

By (2.2) and (2.4), we see that Ψ is surjective and $Ker(\Psi) = \langle[\mathfrak{p}_1], \ldots, [\mathfrak{p}_t]\rangle$. Hence, $\mathcal{C}_D/\mathcal{C}_D^{1-\sigma} \cong \mathcal{C}l_D/\mathcal{C}l_D^{1-\sigma}$, so $r_2^+(D) = r_2(D) = t-1$. Since $Ker(\pi) = Im(\Psi) = \{1, [(\alpha)]_+\}$, from the exact sequences

$$1 \to Ker(\pi) \to \mathcal{C}_D^G \to Ker(\Psi) \to 1,$$

we see that $\mathcal{C}_D^G$ is generated by $[\mathfrak{p}_1]_+, \ldots, [\mathfrak{p}_t]_+, [(\alpha)]_+$. Since Ψ is surjective, the homomorphism $Ker(\pi) \to \mathcal{C}_D/\mathcal{C}_D^{1-\sigma}$ is trivial, so $[(\alpha)]_+ \in \mathcal{C}_D^{1-\sigma}$. When $q \equiv 3 \bmod 4$, we have

$$[(\alpha)]_+ = [(\sqrt{D})]_+ = [\mathfrak{p}_1]_+ \cdots [\mathfrak{p}_t]_+$$

by (2.3), so $\mathcal{C}_D^G$ is generated by $[\mathfrak{p}_1]_+, \ldots, [\mathfrak{p}_t]_+$.

Case 2. D is non-special. In this case, since $\mathcal{C}l_D^G$ is generated by $[\mathfrak{p}_1], \ldots, [\mathfrak{p}_t]$, by (2.2), Ψ is trivial, so we have exact sequence

$$1 \to Ker(\pi) \to \mathcal{C}_D^G \to \mathcal{C}l_D^G \to 1.$$

Hence, $r_2^+(D) = r_2(D)+1 = t-1$ and $\mathcal{C}_D^G$ is generated by $[\mathfrak{p}_1]_+, \ldots, [\mathfrak{p}_t]_+$, $[(\beta)]_+$, where $Ker(\pi) = \{1, [(\beta)]_+\}$. Since Ψ is trivial, the homomorphism $Ker(\pi) \to \mathcal{C}_D/\mathcal{C}_D^{1-\sigma}$ is injective, so $[(\beta)]_+ \notin \mathcal{C}_D^{1-\sigma}$. When $q \equiv 3 \bmod 4$, we have

$$[(\beta)]_+ = [(\sqrt{D})]_+ = [\mathfrak{p}_1]_+ \cdots [\mathfrak{p}_t]_+$$

by (2.3), so $\mathcal{C}_D^G$ is generated by $[\mathfrak{p}_1]_+, \ldots, [\mathfrak{p}_t]_+$. □

2.2. Narrow genus field

Let H_D^+ be the *narrow Hilbert class field* of k_D and G_D^+ be the *narrow genus field* of k_D/k. Then we have canonical isomorphisms $\mathcal{C}_D \cong Gal(H_D^+/k_D)$ and $\mathcal{C}_D/\mathcal{C}_D^{1-\sigma} \cong Gal(G_D^+/k_D)$ via the Artin maps. Note that $\mathcal{C}_D^{1-\sigma} = \mathcal{C}_D^2$, $\mathcal{C}_D^G = {}_2\mathcal{C}_D$ and $\mathcal{C}_D/\mathcal{C}_D^{1-\sigma} = \mathcal{C}_D/\mathcal{C}_D^2 \cong {}_2\mathcal{C}_D$, since σ acts on $\mathcal{C}_D$ as -1.

Proposition 2.3. *Let $D \in \mathcal{D}$ with monic irreducible factorization $D = P_1 \cdots P_t$. Then we have $G_D^+ = k(\sqrt{P_1^*}, \ldots, \sqrt{P_t^*})$, where $P_i^* = (-1)^{\deg P_i} P_i$.*

Proof. Note that the narrow Hilbert class field H_D^+ of k_D can be characterized as the maximal abelian extension of k_D which is unramified at all finite primes and which is contained in $\mathbb{F}_q((\sqrt{-1/T}))$. Hence, $k(\sqrt{P_i^*})$ is contained in H_D^+ for all $1 \le i \le t$. Since $k(\sqrt{P_1^*}), \ldots, k(\sqrt{P_t^*})$ are mutually disjoint over k, the result follows from Lemma 2.2. □

2.3. Rèdei-matrix

Let $D \in \mathcal{D}$ with monic irreducible factorization $D = P_1 \cdots P_t$. For $1 \le i \ne j \le t$, let $m_{ij} \in \mathbb{F}_2$ be defined by $(-1)^{m_{ij}} = (\frac{P_i^*}{P_j})$. We define a $t \times t$ or $t \times (t+1)$ matrix R_D over $\mathbb{F}_2$, which is so called the Rèdei-matrix of k_D, as follows.

(i) When D is special or $q \equiv 3 \bmod 4$, then $R_D = (m_{ij})_{1 \le i,j \le t}$, where the diagonal entries m_{ii} are given by the relation $\sum_{i=1}^{t} m_{ij} = 0$.

(ii) When $q \equiv 1 \bmod 4$ and D is non-special, then R_D is the $t \times (t+1)$ matrix obtained from the $t \times t$ matrix $(m_{ij})_{1 \le i,j \le t}$ given in (i) by adjoining the transpose of $(m_{10}\ m_{20}\ \cdots\ m_{t0})$ in the first column, where m_{i0} is defined by $(-1)^{m_{i0}} = (\frac{P_i^*}{B})$ for a generator B of $\mathcal{N}(\mathfrak{b})$. Here $\mathfrak{b}$ is a square free integral ideal of $\mathcal{O}_D$ such that all the prime ideals dividing it are completely split in k_D/k and $[\mathfrak{b}]_+$ is a member of generators of $\mathcal{C}_D^{1-\sigma}$ as in Lemma 2.2.

Proposition 2.4. *We have* $r_4^+(D) = t - 1 - \operatorname{rank} R_D$.

Proof. We follow the arguments in [5, §3]. Consider the following composite map

$$\Phi : \mathcal{C}_D^G \to \mathcal{C}_D/\mathcal{C}_D^{\sigma-1} \cong Gal(G_D^+/k_D) \hookrightarrow Gal(G_D^+/k), \tag{2.5}$$

where the first map is induced by the inclusion $\mathcal{C}_D^G \subseteq \mathcal{C}_D$, and the isomorphism in the middle is the Artin map. For any $[\mathfrak{b}]_+ \in \mathcal{C}_D^G$, we have

$$\Phi([\mathfrak{b}]_+) = \Big(\frac{G_D^+/k_D}{\mathfrak{b}}\Big).$$

Since Φ is a linear map of $\mathbb{F}_2$-vector spaces and $|Ker(\Phi)| = |\mathcal{C}_D^G \cap \mathcal{C}_D^{\sigma-1}| = |\mathcal{C}_D^2/\mathcal{C}_D^4|$, we have

$$r_4^+(D) = \dim(Ker(\Phi)) = \dim(\mathcal{C}_D^G) - \dim(Im(\Phi)) = r_2^+(D) - \operatorname{rank} R_D,$$

where R_D is a matrix representing Φ. By Proposition 2.3, we have $G_D^+ = K_1 \cdots K_t$ with $K_i = k(\sqrt{P_i^*})$ for $1 \le i \le t$. Choose elements $\sigma_1, \ldots, \sigma_t \in Gal(G_D^+/k)$ such that

$$Gal(G_D^+/K_1 \cdots K_{i-1}K_{i+1} \cdots K_t) = \langle \sigma_i \rangle \quad \text{and} \quad \sigma_i(\sqrt{P_i^*}) = -\sqrt{P_i^*}.$$

Then $Gal(G_D^+/k)$ is generated by $\sigma_1, \ldots, \sigma_t$ and $Gal(G_D^+/k_D)$ is the following $\mathbb{F}_2$-subspace of codimension 1:

$$Gal(G_D^+/k_D) = \Big\{\sigma_1^{k_1} \cdots \sigma_t^{k_t} : \sum_{i=1}^{t} k_i \equiv 0 \bmod 2\Big\}.$$

Case 1. D is special or $q \equiv 3 \bmod 4$. In this case, by Lemma 2.2, we only need to consider $[\mathfrak{p}_1]_+, \dots, [\mathfrak{p}_t]_+$ for the matrix R_D representing Φ. We define $R_D = (m_{ij})_{1 \le i,j \le t}$ by

$$\Phi([\mathfrak{p}_j]_+) = \sigma_1^{m_{1j}} \cdots \sigma_t^{m_{tj}},\ 1 \le j \le t.$$

Then we have

$$\Big(\frac{G_D^+/k_D}{\mathfrak{p}_j}\Big)(\sqrt{P_i^*}) = \Phi([\mathfrak{p}_j])(\sqrt{P_i^*}) = (-1)^{m_{ij}}\sqrt{P_i^*},$$

and on the other hand by definition of the Artin symbol

$$\Big(\frac{G_D^+/k_D}{\mathfrak{p}_j}\Big)(\sqrt{P_i^*}) \equiv (\sqrt{P_i^*})^{q^{\deg P_j}} \equiv \Big(\frac{P_i^*}{P_j}\Big)\sqrt{P_i^*} \bmod \mathfrak{P}_j,$$

where $\mathfrak{P}_j | \mathfrak{p}_j$ is a prime ideal of G_D^+. Hence, for $i \ne j$, we get

$$(-1)^{m_{ij}} = \Big(\frac{P_i^*}{P_j}\Big).$$

The diagonal entries m_{ii} are given by the relation $\sum_{i=1}^t m_{ij} = 0$ defining $Gal(G_D^+/k_D)$. Therefore, we have $r_4^+(D) = t - 1 - \operatorname{rank} R_D$.

Case 2. $q \equiv 1 \bmod 4$ and D is non-special. In this case we should consider the generator $[\mathfrak{b}]_+$ of $\mathcal{C}_D^G$ in Lemma 2.2, since $[\mathfrak{b}]_+ \notin \mathcal{C}_D^{1-\sigma} = \mathcal{C}_D^2$. Since any prime ideal of $\mathcal{O}_D$ which is inert over k is a principal generated by a monic irreducible polynomial, and since the classes of the ramified prime ideals are among the generators of $\mathcal{C}_D^G$, we can assume without loss of generality that $\mathfrak{b}$ is an integral ideal such that all the prime ideals dividing it are completely split in k_D/k. We can also assume that $\mathfrak{b}$ is square free. Write $\mathfrak{b} = \prod_v \mathfrak{q}_v$, where $\mathfrak{q}_v$ is a prime ideal of $\mathcal{O}_D$ which is inert over k. Let Q_v be the monic irreducible polynomial such that $\mathcal{N}(\mathfrak{q}_v) = (Q_v)$ and $B = \prod_v Q_v$. Then we have

$$\Big(\frac{G_D^+/k_D}{\mathfrak{q}_v}\Big)(\sqrt{P_i^*}) = \Big(\frac{P_i^*}{Q_v}\Big)\sqrt{P_i^*},$$

which leads to

$$\Big(\frac{G_D^+/k_D}{\mathfrak{b}}\Big)(\sqrt{P_i^*}) = \Big(\frac{P_i^*}{B}\Big)\sqrt{P_i^*}.$$

Then the matrix $R_D = (m_{ij})_{1 \le i \le t, 0 \le j \le t}$ is defined as follows; m_{ij} for $j > 0$ are defined as in Case 1, and $\Phi([\mathfrak{b}]_+) = \sigma_1^{m_{10}} \cdots \sigma_t^{m_{t0}}$. So m_{i0} is determined by

$$(-1)^{m_{i0}} = \Big(\frac{P_i^*}{B}\Big).$$

Then we have $r_4^+(D) = t - 1 - \operatorname{rank} R_D$. □

3. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. Let $\mathcal{P}(d,t)$ be the set of all monic square free polynomials D in $\mathbb{A}$ with $\deg D = d$ and $\omega(D) = t$. Then, as $d \to \infty$, we have ([6, Theorem 2.1] or [1, (1.2)])

$$|\mathcal{P}(d,t)| = \frac{q^d(\log d)^{t-1}}{(t-1)!d} + O\Big(\frac{q^d(\log d)^{t-2}}{d}\Big). \tag{3.1}$$

Let $\mathcal{P}'(d,t)$ be the subset of $\mathcal{P}(d,t)$ consisting of all $D = P_1 \cdots P_t \in \mathcal{P}(d,t)$ such that $\deg P_1 < \cdots < \deg P_t$. Then, as $d \to \infty$, we have ([6, Proposition 2.2])

$$|\mathcal{P}(d,t) \setminus \mathcal{P}'(d,t)| = o\Big(\frac{q^d(\log d)^{t-1}}{d}\Big). \tag{3.2}$$

For given $d_1, \ldots, d_t \in \{0,1\}$ with $d_1 + \cdots + d_t \equiv 0 \bmod 2$ and $\varepsilon_{ij} \in \{\pm 1\}$ for $1 \le i < j \le t$, write $\mathcal{P}'(d,t;\{d_i\},\{\varepsilon_{ij}\})$ for the subset of $\mathcal{P}'(d,t)$ consisting of all $D = P_1 \cdots P_t \in \mathcal{P}'(d,t)$ such that $\deg P_i \equiv d_i \bmod 2$ for $1 \le i \le t$ and $(\frac{P_i^*}{P_j}) = \varepsilon_{ij}$ for $1 \le i < j \le t$. Then, as $d \to \infty$, we have ([3, Proposition 2.3])

$$|\mathcal{P}'(d,t;\{d_i\},\{\varepsilon_{ij}\})| = 2^{1-\frac{t^2+t}{2}} \cdot \frac{q^d(\log d)^{t-1}}{(t-1)!d} + O\Big(\frac{q^d(\log d)^{t-2}}{d}\Big). \tag{3.3}$$

Now, we assume that $q \equiv 3 \bmod 4$. By definition, $A_{t;d} = \{k_D : D \in \mathcal{P}(d,t)\}$, so $|A_{t;d}| = |\mathcal{P}(d,t)|$. Let $A'_{t;d} = \{k_D : D \in \mathcal{P}'(d,t)\}$ and $A'_{t,r;d} = A'_{t;d} \cap A_{t,r;d}$. Then, by (3.2), we have $|A_{t;d}| \sim |A'_{t;d}|$, so

$$\alpha_t(r) = \lim_{\substack{d\to\infty \\ d:\text{even}}} \frac{|A'_{t,r;d}|}{|A'_{t;d}|}. \tag{3.4}$$

For $d_1, \ldots, d_t \in \{0,1\}$ with $d_1 + \cdots + d_t \equiv 0 \bmod 2$, write $\mathcal{M}_r(d_1, \ldots, d_t)$ for the set of all $t \times t$ matrices $M = (m_{ij})$ over $\mathbb{F}_2$ whose rows sum up to the zero row, of rank $t-1-r$, satisfying, for $i \ne j$, $m_{ij} \ne m_{ji}$ if $d_i = d_j = 1$ and $m_{ij} = m_{ji}$ otherwise. By (3.3), we have

$$\begin{aligned} |A'_{t,r;d}| &= \sum_{\substack{d_1,\ldots,d_t \in \{0,1\} \\ d_1+\cdots+d_t \equiv 0(2)}} \sum_{M=(m_{ij}) \in \mathcal{M}_r(d_1,\ldots,d_t)} |\mathcal{P}'(d,t;\{d_i\},\{m_{ij}\})| \\ &\sim 2^{1-\frac{t^2+t}{2}} \cdot \frac{q^d(\log d)^{t-1}}{(t-1)!d} \sum_{\substack{d_1,\ldots,d_t \in \{0,1\} \\ d_1+\cdots+d_t \equiv 0(2)}} |\mathcal{M}_r(d_1,\ldots,d_t)|. \end{aligned} \tag{3.5}$$

Since $|A_{t;d}| \sim |A'_{t;d}|$, by (3.1), (3.4) and (3.5), we have

$$\alpha_t(r) = 2^{1-\frac{t^2+t}{2}} \sum_{\substack{d_1,\ldots,d_t\in\{0,1\}\\ d_1+\cdots+d_t\equiv 0(2)}} |\mathcal{M}_r(d_1,\ldots,d_t)|.$$

The number of matrices in $\mathcal{M}_r(d_1,\ldots,d_t)$ only depends on the number s of indices i such that $d_i = 1$ (note that s must be an even number). Therefore

$$|\mathcal{M}_r(d_1,\ldots,d_t)| = |\mathcal{M}_r(\underbrace{1,\ldots,1}_{s},\underbrace{0,\ldots,0}_{t-s})|.$$

For simplicity, we write

$$\mathcal{M}_r^{(s)} = \mathcal{M}_r(\underbrace{1,\ldots,1}_{s},\underbrace{0,\ldots,0}_{t-s}).$$

By omitting the sth row/column of each matrix (instead of including the condition that the sum of all rows equals the zero row) in $\mathcal{M}_r^{(s)}$, we see that

$$|\mathcal{M}_r^{(s)}| = N'(t-1, t-1-r, s-1),$$

where $N'(n,a,b)$ denotes the number of $n\times n$ matrices (m_{ij}) over $\mathbb{F}_2$ of rank a such that $m_{ij} \neq m_{ji}$ for all $1 \leq i < j \leq b$, and $m_{ij} = m_{ji}$ otherwise. Thus, we have

$$\alpha_t(r) = 2^{1-\frac{t^2+t}{2}} \sum_{\substack{0\leq s\leq t\\ s:\text{even}}} \binom{t}{s} N'(t-1, t-1-r, s-1).$$

Finally, as in [2, §5], we have

$$\alpha_\infty(r) = \frac{2^{-r(r+1)}\prod_{m=1}^{r}(1-2^{-m})^{-1}(1-2^{-m-1})^{-1}}{\prod_{m=2}^{\infty}(1-2^{-m})} = 2^{-r(r+1)}\frac{(\frac{1}{2})_\infty}{(\frac{1}{2})_r(\frac{1}{2})_{r+1}},$$

which competes the proof.

References

[1] S. Bae and H. Jung, *ℓ-ranks of class groups of function fields*, J. Korean Math. Soc. **49** (2012), no. 1, 49-67.

[2] F. Gerth, III, *The* 4*-class ranks of quadratic fields*, Invent. Math. **77** (1984), no. 3, 489-515.

[3] H. Jung, *Density of class groups of imaginary ℓ-cyclic function fields*, Finite Fields Appl. **17** (2011), no. 3, 286-293.

[4] M. Rosen, *Number theory in function fields*, Graduate Texts in Mathematics **210**, Springer-Verlag, New York, 2002, xii+358.

[5] C. Wittmann, *l-class groups of cyclic function fields of degree l*, Finite Fields Appl. **13** (2007), no. 2, 327-347.

[6] C. Wittmann, *Densities for 4-class ranks of quadratic function fields*, J. Number Theory **129** (2009), no. 10, 2635-2645.

*
Department of Mathematics Education
Chungbuk National University
Cheongju 361-763, Republic of Korea
E-mail: hyjung@chungbuk.ac.kr